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+Permanent Address: Department of Physics, University of California at Los Angeles

QUANTUM PHYSICS LABORATORY
PHYSICAL RESEARCH DIVISION

TRW Space Technology Laboratories
One Space Park
Redondo Beach, California

ABSTRACT

Formulations ~~are~~ presented for the second quantized versions of the field theories which lead to Corben's equations of motion. It ~~is~~ demonstrated that an indefinite metric is required to guarantee positive energies for all the particles, but that otherwise the theories are physically unambiguous. The number and properties of the resulting particles ~~are~~ studied and compared with the conclusions from previous work. Alternative formulations ~~are~~ also discussed.

I. INTRODUCTION

A set of relativistic wave equations have recently been proposed whose solutions yield the mass spectra of free particles and their spins in the range 0, 1/2, 1, 3/2, or 2.[1,2] It has been argued[1,2] that the quantum theories which incorporate these equations connect in the correspondence limit to the classical, relativistic theory of spinning particles derived some time ago by Bhabha and Corben.[3] A few recent reports have been devoted to showing that the quantum mechanical version of this theory--i.e., a set of wave equations--implies a collection of particles whose masses, charges, and spins show an impressive similarity to those of the stable particles and resonances presently observed.

In this paper the structure of the Corben theory is studied in detail. The field theories which lead to these equations are shown to allow a physically consistent second quantization, and the number and properties of the particles which emerge are re-examined from this viewpoint.

The four Corben equations, together with the spins of the associated particles, are

$$\left(\frac{1}{i}\,\gamma_\mu\,\partial_\mu + m + \frac{1}{4}\,m_o\,\sigma_{\mu\nu}\,\sigma'_{\mu\nu}\right)\psi = 0 \qquad \text{(spin 0, 1)} \tag{1}$$

$$\left(\frac{1}{i}\,\gamma_\mu\,\partial_\mu + m + \frac{1}{2}\,m_o\,\sigma_{\mu\nu}\,\beta_{\mu\nu}\right)\psi = 0 \qquad \text{(spin 1/2, 3/2)} \tag{2}$$

$$\left(\frac{1}{i}\,\beta_\mu\,\partial_\mu + m + \frac{1}{2}\,m_o\,\beta_{\mu\nu}\,\sigma_{\mu\nu}\right)\psi = 0 \qquad \text{(spin 1/2, 3/2)} \tag{3}$$

$$\left(\frac{1}{i}\,\beta_\mu\,\partial_\mu + m + m_o\,\beta_{\mu\nu}\,\beta'_{\mu\nu}\right)\psi = 0 \qquad \text{(spin 0, 1, 2)} \tag{4}$$

The γ_μ, γ'_μ, β_μ, β'_μ are all to be considered as acting on separate vector spaces, and hence the dimension of each ψ is the product of the dimensions of the two such operators which occur in the equation. The γ_μ and γ'_μ are four dimensional and fulfill the Dirac anticommutation rules,

$$\left\{\gamma_\mu, \gamma_\nu\right\} = \left\{\gamma'_\mu, \gamma'_\nu\right\} = -2\,\delta_{\mu\nu}, \tag{5}$$

whereas the β_μ and β'_μ are either one, five, or ten dimensional and satisfy the Duffin-Kemmer-Petiau relation typified by[8]

$$\beta_\mu \beta_\nu \beta_\sigma + \beta_\sigma \beta_\nu \beta_\mu = -\beta_\mu \delta_{\nu\sigma} - \beta_\sigma \delta_{\nu\mu} \quad . \tag{6}$$

The γ_μ, γ'_μ, β_μ, β'_μ are all chosen to be antihermitian, and $\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$, $\beta_{\mu\nu} = i\left[\beta_\mu, \beta_\nu\right]$ with similar relations for the primed matrices. Each of the Eqs. (1-4) has two parameters, m and m_o. Although the best "fit" to the observed particle spectra is obtained by giving these parameters somewhat different values in the four cases, we do not make this distinction. Each of the four equations will be discussed separately, and it will always be clear from the context to which equation the parameters refer.

In Sections (II-V) we discuss in turn each of the theories which incorporate Eqs. (1-4). The theory corresponding to Eq. (1) is studied in Sec. II. Two alternative formulations are presented, but the first formulation is considered the more natural; all the solutions to Eq. (1) are

retained, and a parity suggests itself for all the resulting particles. The charges also are determined if the electric current proposed by Corben is employed. In contrast to this approach, we then note in the second formulation that almost none of these conclusions are actually necessary if all we require is the existence of a field ψ satisfying Eq. (1). There exists the freedom, consistent with this requirement, to retain arbitrarily few of the solutions to Eq. (1) and to designate independently the charges and parities of the remaining particles. When this freedom is utilized, however, the field ψ assumes a rather remote role, which is difficult to understand if Eq. (1) is to be the basis of the theory.

In Secs. (III-V) we display the structure of the theories corresponding to Eqs. (2-4) from a viewpoint analogous to the first formulation in Sec. II. The properties of the solutions are presented in detail, and they are shown to be different, in some respects, from what was believed previously. In particular, we shall see that for each mass and spin the multiplicity of the solutions is not in agreement with the observed particle spectrum, and that it will be difficult to rule out only the unwanted solutions in a general manner. However, we emphasize that these theories could also be formulated in analogy with the second approach in Sec. II. If this were done, and the unpleasant feature referred to above were accepted, it would then be possible to retain only those solutions which can be made to correspond to observed particles. From either point of view, the spin of a solution is fixed by the theory.

II. FIELD THEORY OF EQUATION (1)

A. First Formulation

In analogy with the conventional Dirac theory, the field Eq. (1) suggests a Lagrangian density

$$\mathcal{L} = \bar{\psi} \left(\gamma_\mu \frac{1}{i} \partial_\mu + \mathcal{M}_1 \right) \psi \quad , \tag{7}$$

with

$$\mathcal{M}_1 = m + \frac{1}{4} m_o \, \sigma_{\mu\nu} \, \sigma'_{\mu\nu} \quad , \tag{8}$$

and

$$\bar{\psi} = \psi^+ \gamma_o \gamma'_o \quad .$$

From the Lagrangian density in Eq. (7), the expressions for the energy momentum four vector, and for the generalized angular momentum tensor can be derived in the standard manner. There results

$$P_j = : \int d\underline{x} \, \bar{\psi} \, \gamma_o \frac{1}{i} \partial_j \, \psi : \quad , \tag{9}$$

$$H = : \int d\underline{x} \, \bar{\psi} \left(\underline{\gamma} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_1 \right) \psi : \quad , \tag{10}$$

and

$$J_{\mu\nu} = : \int d\underline{x} \, \bar{\psi} \, \gamma_o \left(x_\mu \frac{1}{i} \partial_\nu - x_\nu \frac{1}{i} \partial_\mu + \frac{1}{2} \sigma_{\mu\nu} + \frac{1}{2} \sigma'_{\mu\nu} \right) \psi : \tag{11}$$

The double dots on both sides of these expressions indicate the normal ordered products obtained by moving all destruction operators to the right. From Eq. (11) we see that the spin of the particles contained in the field ψ is $1/2(\underline{\sigma} + \underline{\sigma}')$ and it is therefore either 0 or 1. Complying with the TCP theorem,[9] we postulate the commutation rules

$$\left[\psi(x), \bar{\psi}(y)\gamma_o\right]_{x_o = y_o} = \delta(\underline{x} - \underline{y}) . \tag{12}$$

To see more clearly the decomposition of the field ψ into its normal modes (i.e. particles), we describe the theory in momentum space. Once this is accomplished, it will also be easy to check that all the particles yield a positive contribution to the energy in Eq. (10). We write

$$\psi(x) = \frac{1}{(2\pi)^3} \int d^4 p \, \psi(p) e^{ip\cdot x} , \tag{13}$$

and observe from Eq. (1) that $\psi(p)$ satisfies

$$(\gamma \cdot p + \mathcal{M}_1)\psi(p) = 0 , \tag{14}$$

which, for zero three momentum, reduces to

$$\gamma_o \mathcal{M}_1 \psi(0, W) = W \psi(0, W) . \tag{15}$$

If we multiply Eq. (15) with the matrix $\gamma_o \mathcal{M}_1$, we obtain

$$W^2 \psi(0, W) = m(m + m_o \underline{\sigma} \cdot \underline{\sigma}')\psi(0, W) , \tag{16}$$

so that[1, 6]

$$W^2 = m(m - 3m_o), \qquad (\underline{\sigma} \cdot \underline{\sigma}' = -3, \text{ spin } 0) \tag{17a}$$

$$W^2 = m(m + m_o), \qquad (\underline{\sigma} \cdot \underline{\sigma}' = 1, \text{ spin } 1) \tag{17b}$$

represents the mass spectrum.

In case (17a), there are four linearly independent states corresponding to the four combinations of the signs of W and of the eigenvalue (either $+1$ or -1) of $\gamma_o \gamma_o'$. In case (17b) there are twelve such states which reflect the same alternatives separately for each of the three orientations of the total spin. If we let α indicate the mass and spin orientation and display the sign of $\gamma_o \gamma_o'$ explicitly, we can write

$$\psi(0, W) = \sum_{\alpha} \left\{ \left[a_+(\alpha) u_+(\alpha) + a_-(\alpha) u_-(\alpha) \right] \theta(W) + \left[b_+^+(\alpha) v_+(\alpha) + b_-^+(\alpha) v_-(\alpha) \right] \theta(-W) \right\} \delta(W^2 - m_\alpha^2) \tag{18}$$

where $\theta(W)$ is one for W positive and zero otherwise. The m_α^2 designates either of the two solutions in Eq. (17), and the $a_\pm(\alpha)$ and $b_\pm(\alpha)$ are destruction operators in the Hilbert space for the corresponding particle and antiparticle states.

In terms of $u_\pm(\alpha)$ and $v_\pm(\alpha)$, which describe the particles and antiparticles at rest, we can construct the solutions for arbitrary momenta by applying the appropriate Lorentz transformations. That is, defining

$$u_\pm(\underline{p}, \alpha) \equiv \exp\left[\frac{1}{2} \hat{p}_i \, (\sigma_{i4} + \sigma_{i4}') \, \theta_\alpha(p) \right] u_\pm(\alpha), \tag{19a}$$

$$v_\pm(\underline{p}, \alpha) \equiv \exp\left[\frac{1}{2} \hat{p}_i \, (\sigma_{i4} + \sigma_{i4}') \, \theta_\alpha(p) \right] v_\pm(\alpha), \tag{19b}$$

where $\theta_\alpha(p) \equiv \sinh^{-1} p/m_\alpha$ and $\hat{p} = \underline{p}\,|\underline{p}|^{-1}$, there follows

$$(\gamma \cdot p_\alpha + \mathcal{m}_1) u_\pm(\underline{p}, \alpha) = 0 \qquad (-\gamma \cdot p_\alpha + \mathcal{m}_1) v_\pm(\underline{p}, \alpha) = 0 \ , \tag{20}$$

where $p_\alpha = \left(\underline{p}, \sqrt{\underline{p}^2 + m_\alpha^2}\right)$. We can now write

$$\begin{aligned}\psi(p, p_o) = \sum_\alpha \Big\{ & \left[a_+(\underline{p}, \alpha) u_+(\underline{p}, \alpha) + a_-(\underline{p}, \alpha) u_-(\underline{p}, \alpha)\right] \theta(p_o) \\ & + \left[b_+^+(-\underline{p}, \alpha) v_+(-\underline{p}, \alpha) + b_-^+(-\underline{p}, \alpha) v_-(-\underline{p}, \alpha)\right] \theta(-p_o) \Big\} \delta(p^2 - m_\alpha^2) \ ,\end{aligned} \tag{21}$$

and by substituting this expression into Eq. (13)

$$\begin{aligned}\psi(x) = \frac{1}{(2\pi)^3} \sum_\alpha \int \frac{d\underline{p}}{2\,\omega_\alpha(p)} \Big\{ & \left[a_+(\underline{p}, \alpha) u_+(\underline{p}, \alpha) + a_-(\underline{p}, \alpha) u(\underline{p}, \alpha)\right] e^{ip \cdot x} \\ & + \left[b_+^+(\underline{p}, \alpha) v_+(\underline{p}, \alpha) + b_-^+(\underline{p}, \alpha) v_-(\underline{p}, \alpha)\right] e^{-ip \cdot x} \Big\} \ ,\end{aligned} \tag{22}$$

where $\omega_\alpha(p) = \left(\underline{p}^2 + m_\alpha^2\right)^{1/2}$.

The orthogonality properties of the $u_\pm(p, \alpha)$ and $v_\pm(p, \alpha)$ are derived in the Appendix. With a convenient normalization, they can be expressed as

$$\begin{aligned} u_-^+(\underline{p}, \alpha)\, \gamma_o'\, u_-(\underline{p}, \alpha') &= 0 \ , \qquad u_+^+(\underline{p}, \alpha)\, \gamma_o'\, v_\pm(-\underline{p}, \alpha') = 0, \\ v_-^+(\underline{p}, \alpha)\, \gamma_o'\, v_-(\underline{p}, \alpha') &= 0 \ , \qquad u_-^+(\underline{p}, \alpha)\, \gamma_o'\, v_\pm(-\underline{p}, \alpha') = 0, \end{aligned} \tag{23a}$$

and

$$\begin{aligned} u_\pm^+(\underline{p}, \alpha)\, \gamma_o'\, u_\pm(\underline{p}, \alpha') &= \pm\, 2\, \omega_\alpha(p)\, \delta_{\alpha\alpha'} \ , \\ v_\pm^+(\underline{p}, \alpha)\, \gamma_o'\, v_\pm(\underline{p}, \alpha') &= \mp\, 2\, \omega_\alpha(p)\, \delta_{\alpha\alpha'} \ . \end{aligned} \tag{23b}$$

These relations then allow Eq. (22) to be inverted

$$a_{\pm}(\underline{p}, \alpha) = \pm \int d\underline{x} e^{-ipx} u_{\pm}^{+}(\underline{p}, \alpha) \gamma_0' \psi(x) \tag{24a}$$

$$b_{\pm}(\underline{p}, \alpha) = \mp \int d\underline{x} \psi^{+}(x) \gamma_0' v_{\pm}(\underline{p}, \alpha) e^{-ipx} \quad , \tag{24b}$$

so that from Eq. (12) we have the commutation rules

$$\left[a_{\pm}(\underline{p}, \alpha), a_{\pm}^{+}(\underline{p}', \alpha')\right] = \left[b_{\pm}(\underline{p}, \alpha), b_{\pm}^{+}(\underline{p}', \alpha')\right] = \pm 2\omega_{\alpha}(p)(2\pi)^3 \delta_{\alpha\alpha'} \delta(\underline{p} - \underline{p}') \tag{25}$$

All other commutators are zero. We note in particular that the operators for the modes with the negative signature (sign of $\gamma_0 \gamma_0'$) demonstrate the "wrong" sign in their commutation rules. Before we discuss this feature, let us look at the expression for the energy in Eq. (10) when expressed in terms of these elementary creation and destruction operators. By substituting Eq. (22) into Eq. (10) and employing the relations (23), we obtain

$$H = \frac{1}{(2\pi)^3} \sum_{\alpha} \int \frac{d\underline{p}}{2\omega_{\alpha}(p)} \left[a_{+}^{+}(\underline{p}, \alpha) a_{+}(\underline{p}, \alpha) + b_{+}^{+}(\underline{p}, \alpha) b_{+}(\underline{p}, \alpha) \right.$$

$$\left. - a_{-}^{+}(\underline{p}, \alpha) a_{-}(\underline{p}, \alpha) - b_{-}^{+}(\underline{p}, \alpha) b_{-}(\underline{p}, \alpha)\right] \omega_{\alpha}(p) \tag{26}$$

We observe that in both Eqs. (25 and (26) the terms involving the modes with the negative signature appear with a sign opposite to what is conventional.[10] In both instances this situation would be corrected if we could simply interpret $a_{-}^{+}(\underline{p}, \alpha)$ and $b_{-}^{+}(\underline{p}, \alpha)$ as not the Hermitian adjoints of $a_{-}(\underline{p}, \alpha)$ and $b_{-}(\underline{p}, \alpha)$, but the negative of the Hermitian adjoints.[11] Formally

this can be realized if we think of the theory as quantized with an indefinite metric. That is, if the usual Hermitian adjoint is denoted by an asterisk (*), we can define

$$\psi^{+} = \eta\ \psi^{*}\ \eta\ , \tag{27}$$

where the metric η is

$$\eta = \eta^{+} = \eta^{-1} = \exp\left\{\frac{\pi i}{(2\pi)^3} \sum_{\alpha} \int \frac{d\underline{p}}{2\ \omega_\alpha(p)} \left[a_{-}^{+}(p, \alpha)\, a_{-}(p, \alpha) + b_{-}^{+}(p, \alpha)\, b_{-}(p, \alpha)\right]\right\} \tag{28}$$

Let us think of the theory as quantized with this indefinite metric, but realize that it is simply a formal device for reinterpreting the (+) adjoint. For the positive signature modes, the (+) adjoint and the Hermitian adjoint are the same.

With this reinterpretation, the theory defined by the Lagrangian density (7) is characterized by a positive definite energy, and with a positive norm for all states. As a free field theory, it is therefore physically consistent.

We now discuss the charges and parities of the particles. We have seen that there are eight particles in the theory, four with spin 0 and four with spin 1. Eq. (17) shows that the masses of the particles depend only on their spin and that the spin 1 particles are the more massive. As has been suggested,[6] the parameters m and m_o can be adjusted to fit the masses of the spin 0 and spin 1 particles to the observed masses of the K and K^* mesons.

The electric current which has been proposed[6] for the theory of Eq. (1) is

$$j_\mu = e : \overline{\Psi} \gamma_\mu \psi : \quad , \tag{29}$$

and the operator for the charge is therefore

$$Q = e : \int d\underline{x} \, \overline{\Psi} \gamma_o \psi \, . \tag{30}$$

In order to make clear the charges carried by the various particles, we express Eq. (30) explicitly in terms of the elementary creation and destruction operators. By decomposing the fields ψ and $\overline{\Psi}$ in Eq. (30) according to Eq. (22), and making use of the relations (23), we obtain

$$Q = \frac{e}{(2\pi)^3} \sum_\alpha \int \frac{d\underline{p}}{2\,\omega_\alpha(p)} \Big[a_+^+ (p, \alpha)\, a_+(p, \alpha) - b_+^+ (p, \alpha)\, b_+ (p, \alpha)$$

$$- a_-^+ (p, \alpha)\, a_- (p, \alpha) + b_-^+ (p, \alpha)\, b_- (p, \alpha) \Big] \quad . \tag{31}$$

The charges carried by all the particles are now apparent, if we compare Eq. (31) with Eq. (26) and remember that all the modes must contribute a positive energy. The particles destroyed by the operators $a_\pm (p, \alpha)$ carry the charge e, whereas those destroyed by the $b_\pm (p, \alpha)$ have the opposite charge. In particular, none of the particles are electrically neutral (if the electric current is given by (29)), and thus it is inconsistent to conclude that the theory (as formulated here) describes the K and K^* mesons.

If space inversion (i.e. parity) is to be represented by a simple transformation of the field ψ, the natural choice which suggests itself is

$$P: \ \psi(\underline{x}, t) \rightarrow \gamma_0 \gamma_0' \ \psi(-\underline{x}, t) \ . \tag{32}$$

From Eq. (24), it follows from this definition that

$$P: \begin{array}{l} a_{\pm}(\underline{p}, a) \rightarrow \pm\, a_{\pm}(-\underline{p}, a) \\ \\ b_{\pm}(\underline{p}, a) \rightarrow \pm\, b_{\pm}(-\underline{p}, a) \end{array} \ , \tag{33}$$

and the eight particles divide into two scalars (these with positive signature), two pseudoscalars, two vectors (with negative signature), and two pseudovectors. As before, these conclusions do not allow the theory, as it stands, to describe the K and K^*.

B. <u>Alternative Formulation of the Theory of Equation (1)</u>

We now construct the theory described by Eq. (1) in a manner which will make more clear the connection between the formulation in part A and the conventional field theory of particles with spin 0 and 1. We consider Eq. (1) as the requisite feature of the theory and see what freedom is allowed.

Let us write the field ψ as $\psi_{\alpha\beta}$, a four by four matrix, where the unprimed Dirac operators act on the first subscript and the primed operators act on the second. It can readily be verified, after considerable algebra, that if we define

$$\psi(x) \equiv 1/4 \Big[\sqrt{\frac{m}{2}}\, \phi(x) + \sqrt{\frac{m}{2}}\, \theta(x)\, \gamma_5$$

$$- \sqrt{\frac{m+m_o}{2}} \left(V_\mu(x) - \frac{i}{m}\, \partial_\mu\, \phi(x)\right) \gamma_\mu$$

$$+ \sqrt{\frac{m+m_o}{2}} \left(A_\mu(x) + \frac{i}{m}\, \partial_\mu\, \theta(x)\right) \gamma_\mu\, \gamma_5$$

$$- \sqrt{\frac{1}{8(m+m_o}} \left(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)\right) + \epsilon_{\mu\nu\alpha\beta}\, \partial_\alpha A_\beta(x)\, \sigma_{\mu\nu} \Big], \tag{34}$$

so that

$$\phi(x) = \sqrt{\frac{2}{m}}\, \text{Trace} \left[\psi\, \sigma_2 \right] \tag{35a}$$

$$\theta(x) = \sqrt{\frac{2}{m}}\, \text{Trace} \left[\psi\, \sigma_2\, \gamma_5 \right] \tag{35b}$$

$$V_\mu(x) = \sqrt{\frac{2}{m+m_o}}\, \text{Trace} \left[\psi\, \sigma_2\, \gamma_\mu\, \gamma_5 + \frac{i}{m}\, \partial_\mu\, \psi\, \sigma_2 \right] \tag{35c}$$

$$A_\mu(x) = \sqrt{\frac{2}{m+m_o}}\, \text{Trace} \left[\psi\, \sigma_2\, \gamma_\mu\, \gamma_5 - \frac{i}{m}\, \partial_\mu\, \psi\, \sigma_2\, \gamma_5 \right], \tag{35d}$$

then, to within four divergences, the Lagrangian in Eq. (7) is equal to

$$\mathcal{L} = -1/2 \left[\partial_\mu \phi^+ \partial_\mu \phi + m(m - 3m_o)\, \phi^+ \phi \right]$$

$$+ 1/2 \left[\partial_\mu \theta^+ \partial_\mu \theta + m(m - 3m_o)\, \theta^+ \theta \right]$$

$$+ \Big[1/4\, (\partial_\mu V_\nu^+ - \partial_\nu V_\mu^+)(\partial_\mu V_\nu - \partial_\nu V_\mu)$$

$$+ 1/2\, m(m + m_o)\, V_\mu^+ V_\mu \Big]$$

$$- \Big[1/4\, (\partial_\mu A_\nu^+ - \partial_\nu A_\mu^+)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$+ 1/2\, m(m + m_o)\, A_\mu^+ A_\mu \Big]. \tag{36}$$

Except for the signs[10] of the terms involving the fields θ and V_{μ}, this is a conventional Lagrangian for four uncoupled fields, two of spin 0 and two of spin 1, whose masses agree with Eq. (17).

An alternative point of view can now be adopted which still allows Eq. (1) to be satisfied. We take the Lagrangian as given by Eq. (36), but with the terms involving θ and V_{μ} changed in sign. These changes of sign have the same effect as the introduction of the indefinite metric in part A. The equations of motion are not altered, and hence the field ψ as defined in Eq. (34) still satisfies Eq. (1). In fact, if all we require is Eq. (1), we can ignore in Eq. (36) the terms involving as many of the fields ϕ, θ, etc. as we please, providing that ψ in Eq. (34) is expressed only in terms of the retained fields. It is also clear that the charges and parities of the remaining particles can be fixed arbitrarily. The electric current and the parity transformations would then not necessarily be given by Eq. (1). In particular, it is possible to assign the charges and the parities of the eight particles described by Eq. (36) to conform to those of the K and K^* mesons.

III. FIELD THEORY OF EQUATION (2)

Here we formulate the field theory of Eq. (2) in analogy with part A of Sec. II. The Lagrangian is given by

$$\mathcal{L} = - \bar{\psi} (\gamma_\mu \frac{1}{i} \partial_\mu + \mathcal{M}_2) \psi \ , \tag{37}$$

with

$$\mathcal{M}_2 \equiv m + \frac{1}{2} m_o \sigma_{\mu\nu} \beta_{\mu\nu} \tag{38}$$

$$\bar{\psi} = \psi^+ \gamma_o \eta_4$$

and

$$\eta_\mu = -1 - 2\beta_\mu^2 \ (\mu = 1, 2, 3, 4) \ . \tag{39}$$

The operators $\sigma_{\mu\nu}$ and $\beta_{\mu\nu}$ have beer defined in Sec. I. From Eq. (37) we obtain the expressions for the energy-momentum four vector and for the relativistic angular momentum tensor

$$P_j = : \int d\underline{x} \ \bar{\psi} \gamma_o \frac{1}{i} \partial_j \psi : \tag{40}$$

$$H = : \int d\underline{x} \ \bar{\psi} (\underline{\gamma} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_2) \psi : \tag{41}$$

$$J_{\mu\nu} = : \int d\underline{x} \ \bar{\psi} \gamma_o (x_\mu \frac{1}{i} \partial_\nu - x_\nu \frac{1}{i} \partial_\mu + \frac{1}{2} \sigma_{\mu\nu} + \beta_{\mu\nu}) \psi : \tag{42}$$

If, in analogy with the conventional definition of $\underline{\sigma}$, we define

$$\Sigma_i = \epsilon_{ijk} \beta_{jk} \ \text{(no sum)}, \tag{43}$$

then the particle spins are given by

$$\underline{S} = \frac{1}{2}\,\underline{\sigma} + \underline{\Sigma}\ . \tag{44}$$

From Eqs. (6) and (43), it follows that $\Sigma^2 (\Sigma^2 - 2) = 0$, and hence the Σ spin can be either 0 or 1. The total spin in Eq. (44) is therefore either 1/2 or 3/2, and we thus choose the anticommutation rule[9]

$$\left\{ \psi(x),\ \overline{\Psi}(y)\,\gamma_o \right\}_{x_o = y_o} = \delta(\underline{x} - \underline{y})\ . \tag{45}$$

We again describe the theory in momentum space in order to check that all of the particles yield a positive contribution to the energy, and to observe more clearly the features of the implied particles. For the $\underline{p} = 0$ Fourier components, we have in analogy with Eq. (15)

$$\gamma_o\, \mathcal{M}_2\, \psi(0,\ W) = W\, \psi(0,\ W). \tag{46}$$

The solutions to Eq. (46) are listed in Table 1 of reference 6. As in Sec. II, we denote the positive and negative W solutions of Eq. (46) by $u_{\pm}(\alpha)$ and $v_{\pm}(\alpha)$. Here α refers to the choice of mass $|W|$, and to the choice of spin and its orientation--both of which can be diagonalized simultaneously with the operator $\gamma_o \mathcal{M}_2$ in Eq. (46). The $\pm$ sign refers to the eigenvalue of $\gamma_o\, \eta_4$. For each mass and spin, there are solutions corresponding to both the eigenvalues +1 and -1 of this operator.

As in Sec. II, the wave functions $u_{\pm}(p, a)$ and $v_{\pm}(p, a)$ for finite three momentum can be constructed from the $u_{\pm}(a)$ and $v_{\pm}(a)$ by Lorentz transformation,

$$u_{\pm}(p, a) = \exp\left[\hat{p}_i \left(\tfrac{1}{2}\sigma_{i4} + \beta_{i4}\right)\theta_a(p)\right]u_{\pm}(a) \tag{47a}$$

$$v_{\pm}(p, a) = \exp\left[\hat{p}_i \left(\tfrac{1}{2}\sigma_{i4} + \beta_{i4}\right)\theta_a(p)\right]v_{\pm}(a) \quad , \tag{47b}$$

and hence

$$(\gamma \cdot p_a + \mathcal{m}_2)u_{\pm}(\underline{p}, a) = 0 \qquad (-\gamma \cdot p_a + \mathcal{m}_2)v_{\pm}(\underline{p}, a) = 0. \tag{48}$$

The unit vector $\hat{p}$ and the angle $\theta_a(p)$ are defined after Eq. (19). We could now write the general solution of Eq. (2) in the form of Eq. (22), except that here the sum on a would refer to the masses and spin appropriate to Eq. (2).

From the Appendix, we have the orthonormality relations

$$u_+^{+}(\underline{p}, a)\,\eta_4\,u_-(\underline{p}, a') = 0 \ , \qquad u_+^{+}(\underline{p}, a)\,\eta_4\,v_{\pm}(-\underline{p}, a') = 0 \ ,$$

$$v_+^{+}(\underline{p}, a)\,\eta_4\,v_-(\underline{p}, a') = 0 \ , \qquad u_-^{+}(\underline{p}, a)\,\eta_4\,v_{\pm}(-\underline{p}, a') = 0 \ . \tag{49a}$$

and

$$u_{\pm}^{+}(\underline{p}, a)\,\eta_4\,u_{\pm}(\underline{p}, a') = \pm\, 2\,\omega_a(p)\,\delta_{a a'}$$

$$v_{\pm}^{+}(\underline{p}, a)\,\eta_4\,v_{\pm}(\underline{p}, a') = \mp\, 2\,\omega_a(p)\,\delta_{a a'} \quad . \tag{49b}$$

The equivalent of Eq. (22) can now be inverted to yield

$$a_{\pm}(\underline{p}, \alpha) = \pm\int d\underline{x}\, e^{-ipx}\, u_{\pm}^{+}(\underline{p}, \alpha)\, \eta_4 \psi(x)$$

$$b_{\pm}(\underline{p}, \alpha) = \mp\int d\underline{x}\, \psi^{+}(x)\, \eta_4\, v_{\pm}(\underline{p}, \alpha)\, e^{-ipx} \quad ,$$

so that from Eqs. (45), (49), and (50)

$$\left\{a_{\pm}(\underline{p}, \alpha),\ a_{\pm}^{+}(\underline{p}', \alpha')\right\} = \left\{b_{\mp}(\underline{p}, \alpha),\ b_{\mp}^{+}(\underline{p}', \alpha')\right\} =$$

$$= \pm\, 2\, \omega_{\alpha}(p)(2\pi)^3\, \delta_{\alpha\alpha'}\, \delta(\underline{p} - \underline{p}') \quad .$$

All other pairs of these operators anticommute. Observe, that due to the anti-commutation rules, the relationship between the sign of the anti-commutator and the signature of $\gamma_o \eta_4$ differs for the operators $b_{\pm}(\underline{p}, \alpha)$ and $b_{\pm}^{+}(\underline{p}, \alpha)$ from the corresponding expression in Eq. (25).

Next, we employ the equivalent of Eq. (22) and Eqs. (41, (48), and (49) to obtain

$$H = \frac{1}{(2\pi)^3}\sum_{\alpha}\int \frac{d\underline{p}}{2\omega_{\alpha}(p)}\Big[a_{+}^{+}(\underline{p}, \alpha)\, a_{+}(\underline{p}, \alpha) + b_{-}^{+}(\underline{p}, \alpha)\, b_{-}(\underline{p}, \alpha)$$

$$- a_{-}^{+}(\underline{p}, \alpha)\, a_{-}(\underline{p}, \alpha) - b_{+}^{+}(\underline{p}, \alpha)\, b_{+}(\underline{p}, \alpha)\Big]\, \omega_{\alpha}(p) \quad .$$

We note in this case that the terms involving $a_{-}(\underline{p}, \alpha)$ and $b_{+}(\underline{p}, \alpha)$ occur with the wrong sign in Eqs. (51) and (52). As in Sec. II, we resolve this difficulty by reinterpreting the (+) adjoint of these operators to be the

negative of the (*) Hermitian adjoint.[10, 11] Formally, this can be accomplished by employing the definition (27) with η given by

$$\eta = \exp\left\{\frac{\pi i}{(2\pi)^3}\sum_a \int \frac{d\underline{p}}{2\,\omega_a(p)}\left[a_-^+(\underline{p}, a)\, a_-(\underline{p}, a) + b_+^+(\underline{p}, a)\, b_+(\underline{p}, a)\right]\right\} \tag{53}$$

Once this reinterpretation is understood, all the particles yield a positive contribution to the energy in Eq. (52). To this extent the theory is physically admissible.

The electric current proposed by Corben[6] is

$$j_\mu = \pm \frac{e}{2} : \bar{\psi}\, \gamma_\mu\, (1 + \eta_5)\, \psi : \tag{54}$$

where $\eta_5 = \eta_1\, \eta_2\, \eta_3\, \eta_4$, and the $\pm$ sign depends only upon the representation of the β_μ. Each of the solutions to Eq. (46), and therefore the $u_\pm(\underline{p}, a)$ and $v_\pm(\underline{p}, a)$ defined in Eq. (47) are eigenstates of η_5. We denote by $\delta_a = \pm 1, 0$ the eigenvalue of $\pm 1/2(1 + \eta_5)$ and write the operator for the total charge,

$$Q = \pm \frac{1}{2}\, e : \int d\underline{x}\, \bar{\psi}\, \gamma_o\, (1 + \eta_5)\, \psi : \tag{55}$$

in the form

$$Q = \frac{e}{(2\pi)^3}\sum_a \delta_a \int \frac{d\underline{p}}{2\,\omega_a(p)}\Big[a_+^+(\underline{p}, a)\, a_+(\underline{p}, a) - b_-^+(\underline{p}, a)\, b_-(\underline{p}, a) -$$

$$- a_-^+(\underline{p}, a)\, a_-(\underline{p}, a) + b_+^+(\underline{p}, a)\, b_+(\underline{p}, a)\Big] \tag{56}$$

By comparing Eq. (56) with Eq. (52), and remembering that all the modes yield a positive contribution to the energy, it is evident that the $a_{\pm}(\underline{p}, a)$ destroy particles of charge $e\,\delta_a$ and the $b_{\pm}(\underline{p}, a)$ destroy particles of charge $-e\,\delta_a$.

We have already mentioned that all the solutions are composed of doublets, both members of which are characterized by the same mass and spin but by different eigenvalues (either +1 or -1) of $\gamma_o \eta_4$. This situation is analogous to the occurrence of both signs $\gamma_o \gamma_o'$ in the theory discussed in Sec. II. In that case, the natural interpretation was that every particle had its couterpart differing only in parity. A similar interpretation suggests itself here. Under space inversion, the field ψ would then transform as

$$P : \psi(\underline{x}, t) \rightarrow \gamma_o \eta_4 \psi(-\underline{x}, t) \quad . \tag{57}$$

It has been suggested that this theory describes the nucleons, the Ξ particles, and the N^*_{13} pion-nucleon resonances.[6]. This definition of the parity would suggest, that for every one of these particles, there should exist another with the same mass and spin but with opposite parity. Since such counterparts apparently do not exist, we must conclude that half of the solutions are unaccounted for physically.

Finally, let us emphasize that the theory discussed in this Section can be formulated in a manner analogous to part B in Sec. II. It is thus possible to satisfy Eq. (2), retaining only half its solutions and adjusting arbitrarily the charges and parities of the retained particles to conform to the nucleons, the Ξ baryons, and the N^*_{13} resonances.

IV. FIELD THEORY OF EQUATION (3)

The field Eq. (3) follows from the Lagrangian

$$\mathcal{L} = - \overline{\Psi} \, (\beta_\mu \frac{1}{i} \partial_\mu + \mathcal{M}_3) \, \psi \tag{58}$$

with

$$\mathcal{M}_3 = m + \frac{1}{2} m_o \, \beta_{\mu\nu} \, \sigma_{\mu\nu} \tag{59}$$

and $\overline{\Psi}$ the same as in Eq. (38). The energy-momentum four vector is

$$P_j = : \int d\underline{x} \, \overline{\Psi} \, \beta_o \frac{1}{i} \partial_j \, \psi : \tag{60}$$

$$H = : \int d\underline{x} \, \overline{\Psi} \, (\underline{\beta} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_3) \, \psi : \tag{61}$$

and the relativistic angular momentum tensor is

$$J_{\mu\nu} = : \int d\underline{x} \, \overline{\Psi} \, \beta_o \, (x_\mu \frac{1}{i} \partial_\nu - x_\nu \frac{1}{i} \partial_\mu + \beta_{\mu\nu} + \frac{1}{2} \sigma_{\mu\nu}) \, \psi : \; . \tag{62}$$

As in Sec. III, the spin matrices are given by Eq. (44), and the theory describes particles of spin 1/2 and 3/2.

In Eq. (3) a new feature appears. The coefficient of the time derivative is the singular matrix β_o, and as a consequence, the solutions of this equation do not constitute a complete set. At each instant, the quantity $(\underline{\beta} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_3) \psi$ must be orthogonal to the subspace belonging to the null eigenvalue of β_o. That is, it must be an eigenstate of β_o^2 corresponding to the eigenvalue +1. The anticommutation rules satisfied by ψ and $\overline{\Psi}$

must be constructed to allow arbitrary variations of only those fields which satisfy this constraint. We therefore require

$$\left\{ (\underline{\beta} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_3) \psi(x), \bar{\Psi}(y) \beta_0 \right\}_{x_0 = y_0} = \beta_0^2 (\underline{\beta} \cdot \frac{1}{i} \underline{\nabla} + \mathcal{M}_3) \, \delta(\underline{x} - \underline{y}) \; . \tag{63}$$

In analogy to our procedure in the two previous theories, we first look at the Fourier transform of Eq. (3) restricted to zero three momentum,

$$\mathcal{M}_3 \psi(0, W) = W \beta_0 \psi(0, W) \; . \tag{64}$$

The solutions of Eq. (64) are distinguished by the representation of the β_μ and by the eigenvalues of W, the spin and its orientation, and $\gamma_0 \eta_4$. There are no solutions in the 1×1 representation of the β_μ.

In the 5×5 representation, there are solutions corresponding to four particles of spin 1/2 and with mass $m_a = |W|$ given by[6]

$$W = \pm m \sqrt{\frac{(1+b)(1-3b)}{1-2b}} \quad , \tag{65}$$

where $b = m_0/m$. These solutions differ in the sign of W and in the eigenvalue (either +1 or -1) of $\gamma_0 \eta_4$. In this representation of the β_μ, the solutions to Eq. (64) or Eq. (3) occupy only eight of the twenty dimensions in the direct product space of the β_μ and γ_μ.

In the 10 x 10 representation of the β_μ, 24 of the 40 dimensions are spanned by solutions of Eq. (3). Eight dimensions correspond to solutions with spin 1/2 and sixteen with spin 3/2. The eigenvalues of $W^{(6)}$ are

$$W = \pm m \sqrt{\frac{(1+b)(1-3b)(1-4b)}{1-2b}} \qquad \text{(spin 1/2)} , \tag{66}$$

and

$$W = \pm m \sqrt{1+2b} \qquad \text{(spin 3/2)} . \tag{67}$$

Both signs of the eigenvalue of $\gamma_o \eta_4$ occur in this case also.

We denote the positive (negative) W solutions of Eq. (40) by $u_{\pm}(a)\left(v_{\pm}(a)\right)$ where the $\pm$ sign indicates the eigenvalue of $\gamma_o \eta_4$ and the symbol a denotes the other distinguishing features of the solutions. The wave functions for arbitrary three momentum can be constructed from these $u_{\pm}(a)$ and $v_{\pm}(a)$ in accordance with Eq. (47), and the field ψ can be expanded in the form of Eq. (22). From the orthogonality rules discussed in the Appendix,

$$\begin{aligned} &u_+^+(\underline{p}, a)\, \gamma_o \beta_o u_-(\underline{p}, a') = 0 , \qquad && u_+^+(\underline{p}, a)\, \gamma_o \beta_o v_{\pm}(-\underline{p}, a') = 0 \\ &v_+^+(\underline{p}, a)\, \gamma_o \beta_o v_-(\underline{p}, a') = 0 , \qquad && u_-^+(\underline{p}, a)\, \gamma_o \beta_o v_{\pm}(-\underline{p}\, a') = 0 \end{aligned} \tag{68a}$$

and

$$\begin{aligned} &u_\pm^+(\underline{p}, a)\, \gamma_o \beta_o u_\pm(\underline{p}, a') = \pm 2\, \omega_a(p)\, \delta_{a a'} \\ &v_\pm^+(\underline{p}, a)\, \gamma_o \beta_o v_\pm(\underline{p}, a') = \mp 2\, \omega_a(p)\, \delta_{a a'} , \end{aligned} \tag{68b}$$

we obtain, by inverting the equivalent of Eq. (22),

$$a_{\pm}(\underline{p}, \alpha) = \pm \int d\underline{x}\, e^{-ipx}\, u_{\pm}^{+}(\underline{p}, \alpha)\, \gamma_{o}\, \beta_{o}\, \psi(x) \tag{69a}$$

$$b_{\pm}(\underline{p}, \alpha) = \mp \int d\underline{x}\, \psi^{+}(x)\, \gamma_{o}\, \beta_{o}\, v_{\pm}(\underline{p}, \alpha)\, e^{-ipx} \tag{69b}$$

From Eqs. (63), (68) and (69), and by integrating by parts and making use of the equations

$$(\beta \cdot p_{\alpha} + \varkappa_{3})\, u_{\pm}(\underline{p}, \alpha) = 0 \;, \qquad (-\beta \cdot p_{\alpha} + \varkappa_{3})\, v_{\pm}(\underline{p}, \alpha) = 0 \;, \tag{70}$$

we obtain

$$\left\{a_{\pm}(\underline{p}, \alpha),\, a_{\pm}^{+}(\underline{p}', \alpha')\right\} = \left\{b_{\mp}(\underline{p}, \alpha),\, b_{\mp}^{+}(\underline{p}', \alpha')\right\} = \pm\, 2\,\omega_{\alpha}(p)(2\pi)^{3}\, \delta_{\alpha\alpha'}\, \delta(\underline{p} - \underline{p}') \tag{71}$$

All other combinations anticommute. Finally, Eq. (61) and the equivalent of Eq.(22) allow us to write the total energy as

$$H = \frac{1}{(2\pi)^{3}} \sum_{\alpha} \int \frac{d\underline{p}}{2\,\omega_{\alpha}(p)} \Big[a_{+}^{+}(\underline{p}, \alpha)\, a_{+}(\underline{p}, \alpha) + b_{-}^{+}(\underline{p}, \alpha)\, b_{-}(\underline{p}, \alpha)$$

$$- a_{-}^{+}(\underline{p}, \alpha)\, a_{-}(\underline{p}, \alpha) - b_{+}^{+}(\underline{p}, \alpha)\, b_{+}(\underline{p}, \alpha) \Big]\, \omega_{\alpha}(p) \tag{72}$$

Eqs. (71) and (72) are the same as Eqs. (51) and (52) in Sec. III. The need for reinterpretation of the (+) adjoint and the method for accomplishing this with the metric in Eq. (53) are applicable here exactly as in Sec. III. All the particles then give positive contributions to the energy in Eq. (72).

The total baryonic current proposed for this theory is

$$b_{\mu} = :\, \overline{\psi}\, \beta_{\mu}\, \psi\, : \;, \tag{73}$$

and hence the baryon number B is

$$B = : \int d\underline{x}\, \overline{\psi}\, \beta_o\, \psi : \quad . \tag{74}$$

By substituting the equivalent of Eq. (22) into Eq. (74), and employing the relations (68), we obtain

$$B = \frac{1}{(2\pi)^3} \sum \int \frac{d\underline{p}}{2\,\omega_\alpha(p)} \Big[a_+^+ (\underline{p}, \alpha)\, a_+ (\underline{p}, \alpha) - b_-^+ (\underline{p}, \alpha)\, b_- (\underline{p}, \alpha)$$

$$- a_-^+ (\underline{p}, \alpha)\, a_- (\underline{p}, \alpha) + b_+^+ (\underline{p}, \alpha)\, b_+ (\underline{p}, \alpha) \Big] \quad . \tag{75}$$

Comparison of Eq. (75) with Eq. (72) reveals that the particles destroyed by the $a_{\pm}$ $(\underline{p}, \alpha)$ carry positive baryonic number, whereas those destroyed by the $b_{\pm}$ $(\underline{p}, \alpha)$ carry negative baryonic number.

The electric current which has been suggested(6) for this theory is $j_\mu = e: \overline{\psi}\, \gamma_5\, \beta_\mu\, \psi :$. Although it is conserved, this form is not a plausible candidate in the second quantized version for two reasons: (1) If, as seems most natural, the field $\psi(x, t) \rightarrow \gamma_o\, \eta_4\, \psi(-x, t)$ under parity, then this current is a pseudovector. Of course this definition of space inversion is not required; the theory discussed here could also be formulated in analogy with part B of Sec. I, and this apparent problem could then be avoided. However, (2), the "charge" $Q = e : \int d\underline{x}\, \overline{\psi}\, \gamma_5\, \beta_o\, \psi :$ is not diagonal in the one particle states of this theory. The particles corresponding to the operators $a_{\pm}$ $(\underline{p}, \alpha)$ and $b_{\pm}$ $(\underline{p}, \alpha)$ would not carry definite amounts of charge, and this feature is of course physically inadmissible.(12)

It has been suggested that the theory of Eq. (3) describes the Λ, the Y_o^* (1405 MeV, spin 1/2), and the Y_{03}^* (1520 MeV, spin 3/2). Although the neutrality of all the particles was concluded from the above definition of the electric current, nothing stops us from retaining this conclusion but rejecting the current.

Once the neutrality of all the particles is adopted, it becomes possible to assert that three of the six particles contained in the theory are those mentioned. The particles corresponding to the other solutions are identical to those in mass and spin, but apparently do not occur physically.

V. FIELD THEORY OF EQUATION (4)

The Lagrangian density is

$$\mathcal{L} = -\bar{\psi}\,(\beta_\mu \frac{1}{i}\,\partial_\mu + \mathcal{M}_4)\,\psi\ , \tag{76}$$

with

$$\mathcal{M}_4 = m + m_o\,\beta_{\mu\nu}\,\beta'_{\mu\nu}\ , \tag{77}$$

and

$$\bar{\psi} = \psi^+\ \eta_4\,\eta'_4 \tag{78}$$

The energy-momentum four vector takes the form

$$P_j = :\int d\underline{x}\,\bar{\psi}\,\beta_o\,\frac{1}{i}\,\partial_j\,\psi\ : \tag{79}$$

$$H = :\int d\underline{x}\,\bar{\psi}\,(\underline{\beta}\cdot\frac{1}{i}\,\underline{\nabla} + \mathcal{M}_3)\,\psi\ : \tag{80}$$

and

$$J_{\mu\nu} = :\int d\underline{x}\,\bar{\psi}\,\beta_o\,(x_\mu\,\frac{1}{i}\,\partial_\nu - x_\nu\,\frac{1}{i}\,\partial_\mu + \beta_{\mu\nu} + \beta'_{\mu\nu})\,\psi\ : \tag{81}$$

We note again that the matrix β_o is singular and that the solutions to Eq. (4) do not constitute a complete set. We can repeat here the arguments leading to Eq. (63), with the exception that since the spin is integral we postulate commutation rules,[9]

$$\left[(\underline{\beta}\cdot\frac{1}{i}\,\nabla + \mathcal{M}_4)\,\psi\,(x),\ \bar{\psi}\,(y)\,\beta_o\right]_{x_o = y_o} - \beta_o^2\,(\underline{\beta}\cdot\frac{1}{i}\,\underline{\nabla} + \mathcal{M}_4)\ \delta\,(\underline{x} - \underline{y}).$$

The Fourier transform of Eq. (4), restricted to zero three momentum, is

$$\mathcal{M}_4 \psi(0, W) = W \beta_o \psi(0, W) \quad . \tag{83}$$

The solutions of Eq. (83) are distinguished by the representations of β_μ and β'_μ and by the eigenvalues of W, of the spin, and of $\eta_4 \eta'_4$ (either +1 or -1). All of these possibilities occur and have been tabulated extensively in references 4, 6, and 7.

If the sign of the eigenvalue of $\eta_4 \eta'_4$ is indicated explicitly, we can write the positive and negative W solutions of Eq. (83) as $u_\pm(a)$ and $v_\pm(a)$. We can then construct the wave functions for arbitrary three momentum,

$$u_\pm(\underline{p}, a) = \exp\left[\hat{p}_i\, (\beta_{i4} + \beta'_{i4})\, \theta_a(p)\right] u_\pm(a) \tag{84a}$$

$$v_\pm(\underline{p}, a) = \exp\left[\hat{p}_i\, (\beta_{i4} + \beta'_{i4})\, \theta_a(p)\right] v_\pm(a)\, , \tag{84b}$$

where $\hat{p}$ and $\theta_a(p)$ have the same meaning as stated after Eq. (19). It follows that

$$(\beta \cdot p_a + \mathcal{M}_4) u_\pm(\underline{p}, a) = 0\, , \quad (-\beta \cdot p_a + \mathcal{M}_4) v_\pm(\underline{p}, a) = 0 \tag{85}$$

As discussed in the Appendix, the orthogonality relations are

$$u_+^+(\underline{p}, a)\, \eta'_4\, \beta_o\, u_-(\underline{p}, a') = 0\, , \qquad u_+^+(\underline{p}, a)\, \eta'_4\, \beta_o\, v_\pm(-\underline{p}, a') = 0\, ,$$

$$v_+^+(\underline{p}, a)\, \eta'_4\, \beta_o\, v_-(\underline{p}, a') = 0\, , \qquad u_-^+(\underline{p}, a)\, \eta'_4\, \beta_o\, v_\pm(-\underline{p}, a') = 0\, , \tag{86a}$$

and

$$u_{\pm}^{+}(\underline{p}, \alpha)\, \eta_4'\, \beta_o\, u_{\pm}(\underline{p}, \alpha') = \pm 2\omega_\alpha(p)\, \delta_{\alpha\alpha'}$$

$$v_{\pm}^{+}(\underline{p}, \alpha)\, \eta_4'\, \beta_o\, v_{\pm}(\underline{p}, \alpha') = \mp 2\omega_\alpha(p)\, \delta_{\alpha\alpha'} \quad . \tag{86b}$$

When the field ψ is decomposed in the form of Eq. (22), the relations (86) can be employed to obtain

$$a_{\pm}(\underline{p}, \alpha) = \pm \int d\underline{x}\, e^{-ipx}\, u_{\pm}^{+}(\underline{p}, \alpha)\eta_4'\, \beta_o\, \psi(x) \tag{87a}$$

$$b_{\pm}(\underline{p}, \alpha) = \mp \int d\underline{x}\, \psi^{+}(x)\, \eta_4'\, \beta_o\, v_{\pm}(\underline{p}, \alpha)\, e^{-ipx} \quad , \tag{87b}$$

from which, making use also of Eqs. (82), (85), and (86),

$$\left[a_{\pm}(\underline{p}, \alpha),\ a_{\pm}^{+}(\underline{p}', \alpha')\right] = \left[b_{\pm}(\underline{p}, \alpha),\ b_{\pm}^{+}(\underline{p}', \alpha')\right]$$

$$= \pm 2\omega_\alpha(p)(2\pi)^3\, \delta_{\alpha\alpha'}\, \delta(\underline{p} - \underline{p}') \tag{88}$$

Finally, by substituting the equivalent of (22) into Eq. (80), the total energy can be written as

$$H = \frac{1}{(2\pi)^3} \sum_\alpha \int \frac{d\underline{p}}{2\omega_\alpha(p)} \Big[a_{+}^{+}(\underline{p}, \alpha)\, a_{+}(\underline{p}, \alpha) + b_{+}^{+}(\underline{p}, \alpha)\, b_{+}(\underline{p}, \alpha)$$

$$- a_{-}^{+}(\underline{p}, \alpha)\, a_{-}(\underline{p}, \alpha) - b_{-}^{+}(\underline{p}, \alpha)\, b_{-}(\underline{p}, \alpha)\Big]\, \omega_\alpha(p)\ . \tag{89}$$

We observe that Eqs. (88) and (89) are the same as Eqs. (25) and (26). The theories of Eqs. (1) and (4) should hence, formally, be quantized in the same manner. The discussion after Eq. (26) and the form of the metric in Eq. (28) apply to both theories.

The electric current which has been proposed[4, 6, 7] for this theory is

$$j_\mu = \frac{1}{2} e : \bar{\psi} \beta_\mu (1 + \eta_5') \psi : \quad . \tag{90}$$

All the $u_{\pm}(\underline{p}, \alpha)$ and $v_{\pm}(\underline{p}, \alpha)$ are eigenstates of η_5' with the eigenvalue +1 or -1, and for a given mass and spin, the four particles corresponding to positive and negative W solutions, and to the two signs of $\eta_4 \eta_4'$, all have the same sign of this eigenvalue. From Eq. (90), by decomposing the fields ψ and $\bar{\psi}$ according to Eq. (22) and employing the orthogonality relations (86), we can write the charge Q as

$$Q = \frac{e}{(2\pi)^3} \sum_\alpha{}' \int \frac{d\underline{p}}{2\,\omega_\alpha(p)} \Big[a_+^+(\underline{p}, \alpha)\, a_+(\underline{p}, \alpha) - b_+^+(\underline{p}, \alpha)\, b_+(\underline{p}, \alpha)$$

$$- a_-^+(\underline{p}, \alpha)\, a_-(\underline{p}, \alpha) + b_-^+(\underline{p}, \alpha)\, b_-(\underline{p}, \alpha) \Big] \quad , \tag{91}$$

where the primed sum includes only those terms with the eigenvalue +1 of η_5'. Comparing Eqs. (91) and (89), we see that of the particles included in the primed sum, those destroyed by the $a_{\pm}(\underline{p}, \alpha)$ have the charge e whereas those destroyed by the $b_{\pm}(\underline{p}, \alpha)$ carry charge -e.

Table 2 of reference 7 displays the essential features of the particle spectrum arising from Eq. (4). It has been emphasized there that the two parameters m and m_o can be adjusted to fit the masses and spins to those of a large number of the observed strangeness zero bosons. The multiplicity of the solutions, however, does not agree with experiment. The theory actually contains four particles for each mass and spin; two corresponding to both eigenvalues of $\eta_4 \eta_4'$, and another factor of two because of both positive and negative frequency solutions.

The experimental situation requires that for charged particles there should be two degenerate solutions corresponding to both signs of the charge. The theory thus contains twice as many particles as are actually observed. For neutral particles no degeneracy is required, and hence there are four times as many solutions as can be accounted for physically. Exceptions to this latter conclusion, reducing the factor of four to a factor of two, occur in two cases when the solutions of this theory are related to the observed particles as in reference 7. There, the ω and ρ_o correspond to the two degenerate, positive frequency solutions which differ only in their eigenvalue of $\eta_4 \eta'_4$. The two negative frequency solutions are hence the only two with this mass and spin which are unaccounted for physically. Similar remarks apply for the f and B_o resonances.

Finally, let us note that if in analogy to the previous theories (see, for example, Eq. (32)), space inversion is given by

$$P: \quad \psi(\underline{x}, t) \rightarrow \eta_4 \eta'_4 \psi(-\underline{x}, t) \tag{92}$$

then the ρ_o and ω would have opposite parities, as would also the f and B_o. Since these conclusions are in contradiction to experiment (at least for the ρ and ω) the definition of parity in Eq. (92) is incompatible with the interpretation of these solutions in reference 7.

VI. SUMMARY

We have seen that the field theories which lead to Corben's equations of motion can be made physically consistent, but that the multiplicity of the solutions imply considerably more particles than have been observed. It is possible to adjust the parameters m and m_o to fit the masses and spins to experiment surprisingly well. On the other hand, the number of particles predicted at each such set of values does not agree with the experimental situation.

Our viewpoint has been to take seriously the field theories which lead to Corben's equations of motion and to examine in detail the number and properties of the particles which result. Let us emphasize again that this is not the only possible approach. We could, for example, simply demand that the Corben equations be satisfied without retaining all the solutions. In part B of Sec. II we saw how this procedure could be formalized for the theory of Eq. (1). It is clear that similar methods could be applied to the other equations. We should keep in mind also, that in addition to the four equations discussed here, the Corben point of view actually suggests many more equations reflecting the fact that the term involving m_o can couple together arbitrary spin matrices.[13] If the theories corresponding to these additional equations are studied, most of the solutions will describe particles of high spin ($s > 2$) and with masses above the present experimental observations. There will be some solutions, however, which refer to particles with spins and masses in the range considered in this paper. It seems likely that when

these additional solutions are taken into account, a reinterpretation of the particles predicted by Eqs. (1-4) will be suggested, and it is possible that the multiplicity of the solutions existing then will suggest a simple, plausible scheme for ruling out those which do not fit in the observed particle spectrum.

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APPENDIX

In this Appendix we indicate the derivation of the orthogonality relations employed in Eqs. (23), (49), (68), and (86), which refer, respectively, to the theories of Eqs. (1-4).

A. Equation (23)

It follows from the definition of $u_{\pm}(a)$ and $v_{\pm}(a)$ in Sec. II that

$$\gamma_o m_a u_{\pm}(a) = \mathcal{M}_1 u_{\pm}(a) \quad , \qquad \gamma_o m_a v_{\pm}(a) = -\mathcal{M}_1 v_{\pm}(a)$$

$$\gamma_o' m_a u_{\pm}(a) = \pm \mathcal{M}_1 u_{\pm}(a) \quad , \qquad \gamma_o' m_a v_{\pm}(a) = \mp \mathcal{M}_1 v_{\pm}(a) \quad . \qquad \text{(A. 1)}$$

If we apply the exponential operator in Eq. (19b) to the relations in (A. 1) we obtain after some algebra

$$(\gamma \cdot p_a + \mathcal{M}_1) u_{\pm}(\underline{p}, a) = 0 \quad , \qquad (-\gamma \cdot p_a + \mathcal{M}_1) v_{\pm}(\underline{p}, a) = 0$$

$$(\pm \gamma' \cdot p_a + \mathcal{M}_1) u_{\pm}(\underline{p}, a) = 0 \ , \qquad (\mp \gamma' \cdot p_a + \mathcal{M}_1) v_{\pm}(\underline{p}, a) = 0 \ . \qquad \text{(A. 2)}$$

To prove the orthogonality relation involving $u_+^+(\underline{p}, a)$ and $u_-(\underline{p}, a)$, consider

$$u_+^+(\underline{p}, a) (-\underline{\gamma}' \cdot \underline{p} + \mathcal{M}_1') u_-(\underline{p}, a') = -\omega_{a'}(p) u_+^+(\underline{p}, a) \gamma_o' u_-(\underline{p}, a')$$

$$= \omega_a(p) u_+^+(\underline{p}, a) \gamma_o' u_-(\underline{p}, a') \ , \qquad \text{(A. 3)}$$

where both equalities arise from (A. 3); the first expression from applying

$(-\underline{\gamma}' \cdot \underline{p} + \mathcal{M}_1)$ to the right, and the lower expression form applying it to the left. Clearly, both equalities can only be valid if $u_+^+(\underline{p}, \alpha)\, \gamma_o'\, u_-(\underline{p}, \alpha') = 0$, which is the first of the relations in Eq. (23). The corresponding relation for the negative frequency solutions, $v_+^+(\underline{p}, \alpha)\, \gamma_o'\, v_-(\underline{p}, \alpha') = 0$, can be proved similarly.

Eq. (A. 2) can also easily be employed to prove the orthogonality relations involving $u_+^+(\underline{p}, \alpha)$ and $v_{\pm}(-\underline{p}, \alpha')$. Consider

$$u_+^+(\underline{p}, \alpha)\, \gamma_o \gamma_o' (\underline{\gamma} \cdot \underline{p} + \mathcal{M}_1)\, v_{\pm}(-\underline{p}, \alpha')$$

$$= -\omega_{\alpha'}(p)\, u_+^+(\underline{p}, \alpha)\, \gamma_o'\, v_{\pm}(-\underline{p}, \alpha')$$

$$= \omega_\alpha(p)\, u_+^+(\underline{p}, \alpha)\, \gamma_o'\, v_{\pm}(-\underline{p}, \alpha') \;, \qquad \text{(A. 4)}$$

where again the first equality arises from applyin~ $\cdot\, \underline{p} + \mathcal{M}_1)$ to the right and the second from applying it to the left. It follows obviously that $u_+^+(\underline{p}, \alpha)\, \gamma_o'\, v_{\pm}(-\underline{p}, \alpha') = 0$, which is another of the relations in Eq. (23a). The final orthogonality relation in Eq. (23a) can be proved similarly.

To show that $u_{\pm}^+(\underline{p}, \alpha)\, \gamma_o'\, u_{\pm}(\underline{p}, \alpha') \propto \delta_{\alpha\alpha'}$, we consider

$$u_{\pm}^+(\underline{p}, \alpha)\, \gamma_o \gamma_o' (\underline{\gamma} \cdot \underline{p} + \mathcal{M}_1)\, u_{\pm}(\underline{p}, \alpha')$$

$$= \omega_{\alpha'}(p)\, u_{\pm}^+(\underline{p}, \alpha)\, \gamma_o'\, u_{\pm}(\underline{p}, \alpha')$$

$$= \omega_\alpha(p)\, u_{\pm}^+(\underline{p}, \alpha)\, \gamma_o'\, u_{\pm}(\underline{p}, \alpha'), \qquad \text{(A. 5)}$$

where again the upper and lower terms on the right come from acting with $\underline{\gamma} \cdot \underline{p} + \mathcal{M}_1$, to the right or left, respectively. Except for the factor $\pm 2\omega_\alpha(p)$,

the first version of Eq. (23b) follows from comparing the two forms of (A. 5). That $v_{\pm}^{+}(\underline{p}, a)\, \gamma_o'\, v_{\pm}(\underline{p}, a') \propto \delta_{aa'}$ can be shown analogously.

Finally we must verify the $\pm$ signs appearing in Eq. (23b). Take, for example,

$$u_{\pm}^{+}(\underline{p}, a)\, \gamma_o'\, u_{\pm}(\underline{p}, a)$$

$$= u_{\pm}^{+}(a)\, e^{\frac{1}{2}\hat{p}_i(\sigma_{i4} + \sigma_{i4}')\theta_a(p)}\, \gamma_o'\, e^{\frac{1}{2}\hat{p}_i(\sigma_{i4} + \sigma_{i4}')\theta_a(p)}\, u_{\pm}(a), \tag{A.6}$$

where we have made use of Eq. (19). Taking into account the definition of $\theta_a(p)$ after Eq. (19), Eq. (A. 6) can be rewritten as

$$u_{\pm}^{+}(\underline{p}, a)\, \gamma_o'\, u_{\pm}(\underline{p}, a)$$

$$= u_{\pm}^{+}(a)\, \gamma_o' \left(\frac{\omega_a(p) + \sigma_{i4}\, p_i}{m_a}\right) u_{\pm}(a)$$

$$= \omega_a(p)\, m_a^{-1}\, u_{\pm}^{+}(a)\, \gamma_o'\, u_{\pm}(a) \tag{A.7}$$

$$= \pm\, \omega_a(p)\, m_a^{-2}\, u_{+}^{+}(a)\, \mathcal{M}_1\, u_{+}(a)$$

The second version of (A. 7) follows from the first since $\left\{\sigma_{i4}, \gamma_o\gamma_o'\right\} = 0$ and hence σ_{i4} does not connect two states belonging to the same eigenvalue of $\gamma_o\gamma_o'$. The third version follows from the second by (A. 1). The possibility of choosing the normalization of the $u_{\pm}(\underline{p}, a)$ according to the $\pm$ sign in the first form of (23b) is now apparent since $\mathcal{M}_1 = m + \frac{1}{4} m_o \sigma_{\mu\nu} \sigma_{\mu\nu}'$ is a positive definite operator (for the values of m and m_o actually employed).

This positive definiteness can be easily proved (it is obviously true for small enough m_o), but we do not discuss the proof here. It turns out that $\mathcal{M}_1$ is positive unless m_o is large enough to produce complex m_a in Eq. (A.1).

B. Equation (49)

To demonstrate the orthogonality relations involving $u_+ (\underline{p}, a)$ and $v_{\pm} (-\underline{p}, a')$ we make use of Eq. (48) to write

$$u_+^+ (\underline{p}, a)\, \gamma_o\, \eta_4 \,(\underline{\gamma} \cdot \underline{p} + \mathcal{M}_2)\, v_{\pm} (-\underline{p}, a')$$

$$= -\omega_{a'} (p)\, u_+^+ (\underline{p}, a)\, \eta_4\, v_{\pm} (-\underline{p}, a')$$

$$= \omega_a (p)\, u_+^+ (\underline{p}, a)\, \eta_4\, v_{\pm} (-\underline{p}, a') \quad , \qquad \text{(A.8)}$$

where the first and second equalities arise from acting with $(\underline{\gamma} \cdot \underline{p} + \mathcal{M}_2)$ on the right and left, respectively. The desired relation follows immediately, and the corresponding expression involving $u_-^+ (\underline{p}, a)$ is proved in an identical manner.

The proof of Eq. (49b) follows in complete analogy with that of Eq. (23b). Instead of (A.5) we have

$$u_{\pm}^+ (\underline{p}, a)\, \gamma_o\, \eta_4 \,(\underline{\gamma} \cdot \underline{p} + \mathcal{M}_2)\, u_{\pm} (\underline{p}, a')$$

$$= \omega_{a'} (p)\, u_{\pm}^+ (\underline{p}, a)\, \eta_4\, u_{\pm} (\underline{p}, a')$$

$$= \omega_a (p)\, u_{\pm}^+ (\underline{p}, a)\, \eta_4\, u_{\pm} (\underline{p}, a') \quad , \qquad \text{(A.9)}$$

which demonstrates that the left side of Eq. (49b) is proportional to $\delta_{c\,a'}$. To show that the $u_{\pm}(\underline{p}, a)$ can be normalized according to Eq. (49b), consider

$$u_{\pm}^{+}(\underline{p}, a)\,\eta_4\,u_{\pm}(\underline{p}, a)$$

$$= u_{\pm}^{+}(a)\,e^{\hat{p}_i(\frac{1}{2}\sigma_{i4}\;\beta_{i4})\,\theta_a(p)}\,\eta_4\,e^{\hat{p}_i(\frac{1}{2}\sigma_{i4}+\beta_{i4})\,\theta_a(p)}\,u_{\pm}(a) \tag{A.10}$$

in analogy with (A.6). If γ_0' in Eq. (A.7) is replaced by η_4, the various versions of (A.7) follow identically, and we obtain

$$u_{\pm}^{+}(\underline{p}, a)\,\eta_4\,u_{\pm}(\underline{p}, a) = \pm\,\omega_a(p)\,m_a^{-2}\,u_{\pm}^{+}(a)\,\mathcal{M}_2\,u_{\pm}(a)\ . \tag{A.11}$$

The positive definiteness of the operator $\mathcal{M}_2$ (which can easily be proved) then implies our result. The corresponding expression involving $v_{\pm}(\underline{p}, a)$ can be demonstrated similarly.

Finally we must show that $u_{+}^{+}(\underline{p}, a)\,\eta_4\,u_{-}(\underline{p}, a) = 0$, and also that $v_{+}^{+}(\underline{p}, a)\,\eta_4\,v_{-}(\underline{p}, a) = 0$. We note that these relations are trivially true for all the solutions except for the pair corresponding to the proton and the pair corresponding to the N_{13}^{*+} in Table 1 of reference 6, since the two members of every other pair occur in different representations of the β_{μ}. Consider the matrix element of η_4 written in the form of (A.10). Since η_4 anticommutes with β_{i4}, this expression can be written as

$$u_{+}^{+}(\underline{p}, a)\,\eta_4\,u_{-}(\underline{p}, a) = u_{+}^{+}(a)\,\eta_4\,(\omega_a(p) + \sigma_{i4}\,p_i)\,m_a^{-1}\,u_{-}(a)\ . \tag{A.12}$$

The term involving $\omega_a(p)$ does not contribute, since it commutes with $\gamma_o \eta_4$ and hence doesn't connect eigenstates of $\gamma_o \eta_4$ belonging to different eigenvalues. Further, the η_4 can be replaced by γ_o (since $\eta_4^2 = 1$ and $u_+^+(a) = u_+^+(a)\gamma_o \eta_4$) to yield

$$u_+^+(\underline{p}, a)\eta_4 u_-(\underline{p}, a) = p_i m_a^{-1} u_+^+(a)\gamma_o \sigma_{i4} u_-(a)$$

$$= p_i (2m_a)^{-1} u_+^+(a)\left[\gamma_o, \sigma_{i4}\right] u_-(a) \quad . \tag{A.13}$$

We now use the fact that the $u_\pm(a)$ are solutions of Eq. (46) with $W = m_a$ to rewrite (A.13) as

$$u_+^+(\underline{p}, a)\eta_4 u_-(\underline{p}, a) = p_i (2m_a^2)^{-1} u_+^+(a)\left[\mathcal{M}_2, \sigma_{i4}\right] u_-(a) \quad . \tag{A.14}$$

If we define $\Lambda_i \equiv \beta_{i4}$, $\tau_i \equiv \sigma_{i4}$, and employ Σ_i defined in Eq. (43) to express

$$\mathcal{M}_2 = m + \frac{1}{2} m_o \sigma_{\mu\nu} \beta_{\mu\nu} = m + m_o (\underline{\sigma} \cdot \underline{\Sigma} + \underline{\tau} \cdot \underline{\Lambda}) \quad , \tag{A.15}$$

we can make use of the commutation rules

$$\left[\tau_i, \tau_j\right] = i\,\epsilon_{ijk}\sigma_k \quad , \quad \left[\Sigma_i, \Sigma_j\right] = \left[\Lambda_i, \Lambda_j\right] = i\,\epsilon_{ijk}\Sigma_k \quad ,$$

$$\left[\tau_i, \sigma_j\right] = i\,\epsilon_{ijk}\tau_k \quad , \quad \left[\Lambda_i, \Sigma_j\right] = i\,\epsilon_{ijk}\Lambda_k \quad , \tag{A.16}$$

to rewrite (A.14) as

$$u_+^+(\underline{p}, a)\eta_4 u_-(\underline{p}, a) = m_o p_i (2m_a^2)^{-1} u_+^+(a)\left\{\left[\underline{\sigma} \cdot \underline{\Sigma}, \tau_i\right] + \left[\underline{\tau} \cdot \underline{\Lambda}, \tau_i\right]\right\} u_-(a) \quad . \tag{A.17}$$

From (A.16) we can readily verify that $\left[\underline{\tau}\cdot\underline{\Lambda},\ \tau_i\right] = -\left[\underline{\sigma}\cdot\underline{\Sigma},\ \Lambda_i\right]$. Therefore,

$$u_+^{+}(\underline{p}, a)\,\eta_4\,u_-(\underline{p}, a) = m_o p_i (2m_a^2)^{-1}\, u_+^{+}(a)\left[\underline{\sigma}\cdot\underline{\Sigma},\ (\tau_i - \Lambda_i)\right] u_-(a)\ . \tag{A.18}$$

Now it turns out by explicitly looking at the solutions of Eq. (46), that for the two pairs of solutions for which the left side of (A.18) is not trivially zero (the pairs for the proton and for the N_{13}^{*+} in Table 1 of reference 6) the solutions are also eigenstates of the operator $\underline{\sigma}\cdot\underline{\Sigma}$. The commutator in (A.18) does not therefore contribute to the matrix element.

C. Equation (68)

The orthogonality relations involving the $u_{\pm}(\underline{p}, a)$ and the $v_{\pm}(-\underline{p}, a)$ are proved easily as in the two previous cases. We observe, for example that

$$\begin{aligned}
&u_+^{+}(\underline{p}, a)\,\gamma_o\eta_4(\underline{\beta}\cdot\underline{p} + \mathcal{M}_3)\,v_{\pm}(-\underline{p}, a') \\
&= -\omega_{a'}(p)\,u_+^{+}(\underline{p}, a)\,\gamma_o\beta_o\,v_{\pm}(-\underline{p}, a') \\
&= \omega_a(p)\,u_+^{+}(\underline{p}, a)\,\gamma_o\beta_o\,v_{\pm}(-\underline{p}, a')\ ,
\end{aligned} \tag{A.19}$$

where again the first and second forms on the right follow from acting with $(\underline{\beta}\cdot\underline{p} + \mathcal{M}_3)$ on the $v_{\pm}$ and on the u_+, respectively. As before, the desired orthogonality relation is proved by equating the two equivalent forms of (A.19).

To prove that $u_{\pm}^{+}(\underline{p}, \alpha)\, \gamma_o\, \beta_o\, u_{\pm}(\underline{p}, \alpha)$, or the corresponding expressions involving $v_{\pm}(\underline{p}, \alpha)$, is proportional to $\delta_{\alpha\alpha'}$, merely repeat the discussion of (A.9) but with $(\underline{\gamma} \cdot \underline{p} + \mathcal{M}_2)$ replaced by $(\underline{\beta} \cdot \underline{p} + \mathcal{M}_3)$. The proof that the $\pm$ signs in Eq. (68b) can be chosen as shown is also similar to our procedure in the two previous cases. Since $\eta_4 \beta_o = \beta_o$, we can write

$$u_{\pm}^{+}(\underline{p}, \alpha)\, \gamma_o \beta_o\, u_{\pm}(\underline{p}, \alpha)$$
$$= u_{\pm}^{+}(\alpha)\, \gamma_o\, \eta_4\, e^{-\hat{p}_i \left(\frac{1}{2}\sigma_{i4} + \beta_{i4}\right)\theta_\alpha(p)}\, \beta_o\, e^{\hat{p}_i \left(\frac{1}{2}\sigma_{i4} + \beta_{i4}\right)\theta_\alpha(p)}\, u_{\pm}(\underline{p}, \alpha)\,, \tag{A.20}$$

where we have used the fact that $\gamma_o \eta_4$ anticommutes with both σ_{i4} and β_{i4}. Evaluating the product of β_o and the two exponentials in (A.20), and replacing $\gamma_o \eta_4$ by its eigenvalues ± 1, we obtain

$$u_{\pm}^{+}(\underline{p}, \alpha)\, \gamma_o \beta_o\, u_{\pm}(\underline{p}, \alpha) = \pm\, \omega_\alpha^{-1}(p)\, u_{\pm}^{+}(\alpha) \Big(\underline{\beta} \cdot \underline{p} + \beta_o \omega_\alpha(p)\Big)\, u_{\pm}(\alpha)\ . \tag{A.21}$$

Only the second term on the right hand side contributes, since β_i anti-commutes with $\gamma_o \eta_4$ and therefore can't connect two eigenstates of this operator belonging to the same eigenvalue. The second term can be rewritten by realizing that $u_{\pm}(\alpha)$ is a solution of Eq. (64) with $W = m_\alpha$. We obtain

$$u_{\pm}^{+}(\underline{p}, \alpha)\, \gamma_o \beta_o\, u_{\pm}(\underline{p}, \alpha) = \pm\, m_\alpha^{-1}\, u_{\pm}^{+}(\alpha)\, \mathcal{M}_3\, u_{\pm}(\alpha)\ , \tag{A.22}$$

and the positive definiteness of $\mathcal{M}_3$ (which we again do not prove in detail)

assures us that the choice of signs in the first form of Eq. (68b) is correct. The second form can be verified similarly.

Finally, we must prove the orthogonality relation involving the $u_+(\underline{p}, \alpha)$ and $u_-(\underline{p}, \alpha)$. Eq. (A.21) also is valid if the $\pm$ sign on one of the $u_\pm(\alpha)$ are reversed. In that case only the first term on the right of (A.21) contributes (since $\left[\gamma_o \eta_4, \beta_o\right] = 0$), and we have

$$u_\pm^+(\underline{p}, \alpha)\, \gamma_o \beta_o\, u_\mp(\underline{p}, \alpha) = \pm\, u_\pm^+(\alpha)\, \beta_i\, u_\mp(\alpha)\, p_i \omega_\alpha^{-1}(p) \quad . \tag{A.23}$$

We have not been able to find a simple proof that the right hand side of (A.22) vanishes. We have, however, verified that it is zero by calculating straightforwardly with the solutions to Eq. (64).

D. Equation (86)

We prove first the orthogonality relations involving the $u_\pm(\underline{p}, \alpha)$ and the $v_\pm(-\underline{p}, \alpha)$. We note

$$u_+^+(\underline{p}, \alpha)\, \eta_4 \eta'_4\, (\underline{\beta} \cdot \underline{p} + \mathcal{M}_4)\, v_\pm(-\underline{p}, \alpha') = -\omega_{\alpha'}(p)\, u_+^+(\underline{p}, \alpha)\, \eta'_4 \beta_o v_\pm(-\underline{p}, \alpha')$$

$$= \omega_\alpha(p)\, u_+^+(\underline{p}, \alpha)\, \eta'_4 \beta_o v_\pm(-\underline{p}, \alpha'), \tag{A.24}$$

where, as before, the first term results from acting with $(\underline{\beta} \cdot \underline{p} + \mathcal{M}_4)$ to the right, and the second from acting with this operator to the left. Clearly, both forms are compatible only if $u_+^+(\underline{p}, \alpha)\, \eta'_4 \beta_o v_\pm(-\underline{p}, \alpha') = 0$. It is also clear that a similar relation holds with u_+ replaced by u_-.

The proof of Eq. (86b) is essentially identical to our previous procedures. In (A.9) replace γ_o by η_4, $\underline{\gamma} \cdot \underline{p}$ by $\underline{\beta} \cdot \underline{p}$, and $\mathcal{M}_2$ by $\mathcal{M}_4$. We then have immediately that the left sides of (86b) are proportional to $\delta_{\alpha\alpha'}$. The $\pm$ signs in (86b) are also verified as before. If we replace in (A.20), (A.21), and (A.22) γ_o by η'_4, $\frac{1}{2}\sigma_{i4}$ by β'_{i4}, and $\mathcal{M}_3$ by $\mathcal{M}_4$, the signs in (86b) follow as a result of the positive definiteness of $\mathcal{M}_4$.

The proof of the orthogonality relations involving $u_+(\underline{p}, \alpha)$ and $u_-(\underline{p}, \alpha)$ proceeds as in the previous case, i.e. Eq. (68). We can readily obtain that

$$u_{\pm}(\underline{p}, \alpha)\,\eta'_4\,\beta_o\,u_{\mp}(\underline{p}, \alpha) = \pm\, u^{+}_{\pm}(\alpha)\,\beta_1\,u_{\mp}(\alpha)\,p_1 \quad . \tag{A.25}$$

Again, we have not been able to construct a simple proof that the right side of (A.25) is zero, but that this is indeed true can be verified by looking directly at the solutions of Eq. (83).

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11. To conserve probability, any extension of the theory to include interactions would have to be carefully constructed to guarantee that the interaction Hamiltonian was Hermitian (in the (*) sense).
12. We should note, however, that it is possible to redefine the one particle states as linear combinations of the two degenerate modes at each mass in such a way that the wave functions for the new states are eigenfunctions of γ_5. Since the eigenvalues of γ_5 are ± 1, all the particles would then be electrically charged, and it would not be possible to associate this theory with only neutral particles as has been suggested.
13. See, for example, L. Castell, University College (London) preprint.